On the Reidemeister spectrum and the R_∞ property for some free nilpotent groups

E.G. KUKINA AND V. ROMAN'KOV*

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Abstract

We describe the Reidemeister spectrum $Spec_R G$ for $G = N_{rc}$, the free nilpotent group of rank r and class c, in the cases: $r \in \mathbf{N}$ and $c = 1$; $r = 2, 3$ and $c = 2$; $r = 2$ and $c = 3$, and prove that any group N_{2c} for $c \geq 4$ satisfies to the R_∞ property. As a consequence we obtain that every free solvable group S_{2t} of rank 2 and class $t \geq 2$ (in particular the free metabelian group $M_2 = S_{22}$ of rank 2) satisfies to the R_∞ property. Moreover, we prove that any free solvable group S_{rt} of rank $r \geq 2$ and class t big enough also satisfies to the R_∞ property.[1,2]

Contents

1 Introduction

Let G be a group, and $\varphi : G \to G$ be an automorphism of G. One says that the elements $g, f \in G$ are $\varphi-$*twisted conjugated*, denoted by $g \sim_\varphi f$, if and only if

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there exists $x \in G$ such that $(x\varphi)g = fx$. A class of equivalence $[g]_\varphi$ is called the *Reidemeister class* (or the *$\varphi-$conjugacy class of* φ). The number $R(\varphi)$ of Reidemeister classes is called the *Reidemeister number of* φ. We define the *Reidemeister spectrum* of G by

$$Spec_R(G) = \{R(\varphi)|\varphi \in Aut(G)\}. \quad (1)$$

One says that a group G *has the R_∞ property for automorphisms*, denoted by $G \in R_\infty$, if for every automorphism $\varphi : G \to G$ one has $R(\varphi) = \infty$. The class of groups with R_∞ property is very interesting in view of various of applications in Nielsen-Reidemeister fixed point theory, representation theory, dynamic systems, algebraic geometry and so on. See for instances [3], [4], [5], [6], [8], [9], [10], [18], [21], [20], [19].

In the paper by D. Gonçalves and P. Wong [9] mainly devoted to finitely generated nilpotent groups it was shown that any free nilpotent group N_{2c} of rank 2 and class $c \geq 8$ satisfies to R_∞ property. On the other hand the authors of [9] noted that they do not know how to extend their techniques to N_{rc} for $r \geq 3$, and $c \geq 3$. In the paper by V. Roman'kov [16] it was proved that any free nilpotent group N_{rc} of rank $r = 2$ or $r = 3$ and class $c \geq 4r$, or rank $r \geq 4$ and class $c \geq 2r$, satisfies to the R_∞ property. This result provides a purely algebraic proof of the fact that any absolutely free group $F_r,\ r \geq 2$, has the R_∞ property. Note that in [15] this statement was derived by group-geometrical techniques.

In this paper we mainly consider the free nilpotent groups of small class. We obtain the Reidemeister spectrum of a group N_{rc} for $r \in \mathbf{N}$ and $c = 1$ (free abelian case), for $r = 2, 3$ and $c = 2$ ($2-$nilpotent case), for $r = 2$ and $c = 3$ ($3-$nilpotent case). As a main statement we prove that any group N_{2c} for $c \geq 4$ satisfies to the R_∞ property. This result completes the case of rank 2. As a consequence we obtain that every free solvable group S_{2t} of rank 2 and class $t \geq 2$ (in particular the free metabelian group $M_2 = S_{22}$ of rank 2) satisfies to the R_∞ property. Moreover every free solvable group S_{rt} of rank $r \geq 3$ and class t big enough also satisfies to the R_∞ property.

2 Preliminaries

The results of this section in fact are known by paper [9]. Nevertheless we present them for completeness of the paper.

Let G be a finitely generated group, and let C be a central subgroup of G. For any automorphism $\varphi : G \to G$ define a central subgroup

$$L(C, \varphi) = \{c \in C | \exists x \in G : x\varphi = cx\}. \quad (2)$$

It was shown in [16] that any pair of elements $c_1, c_2 \in C$ are $\varphi-$conjugated in G if and only if

$$c_1^{-1}c_2 \in L(C,\varphi). \tag{3}$$

Thus in the case of a finitely generated abelian group A the set of all $\varphi-$conjugacy classes coincides with the set of all cosets A w.r.t. $L(A,\varphi)$. Note that

$$L(A,\varphi) = Im(\varphi - id). \tag{4}$$

Hence

$$R(\varphi) = [A : L(A,\varphi)]. \tag{5}$$

Let $A(r) = \mathbb{Z}^r$ be a free abelian group of rank $r \in \mathbb{N}$, and $\varphi : A(r) \to A(r)$ be any automorphism.

Easy to see that

$$Spec(\mathbb{Z}) = \{2\} \cup \{\infty\}. \tag{6}$$

We claim that for $r \geq 2$ the spectrum is full, i.e.

$$SpecA(r) = \mathbb{N} \cup \{\infty\}. \tag{7}$$

To prove it enough to find for every number $k \in \mathbb{N}$ a matrix $A(r,k) \in GL_r(\mathbb{Z})$ such that $rank(A(r,k) - E) = k$.

If $r = 2$ or $r = 3$ one can take

$$A(2,k) = \begin{pmatrix} -k & 1 \\ 1 & 0 \end{pmatrix}, \tag{8}$$

and

$$A(3,k) = \begin{pmatrix} 1 & k & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \tag{9}$$

respectively.

In general case for odd $r = 2t + 1$ we put

$$A(r,k) = diag(A(3,k), A(2,1), ..., A(2,1)), \tag{10}$$

and for even $r = 2t$

$$A(r,k) = diag(A(2,k), A(2,1), ..., A(2,1)), \tag{11}$$

where $A(2,1)$ repeats $t-1$ times.

It remains to note that $R(id) = \infty$ for any group $A(r)$.

Let N be a finitely generated torsion free nilpotent group of class k. Let

$$\zeta_0 N = 1 < \zeta_1 N < ... < \zeta_{k-1} N < \zeta_k N = N \tag{12}$$

be the upper central series in N. It is well known (see [11], [13]) that all quotients

$$N_i = N/\zeta_i N,\ A_i = \zeta_{i+1}N/\zeta_i N,\ i = 0, 1, ..., k-1, \tag{13}$$

are finitely generated torsion free groups. In particular, every $A_i,\ i = 0, 1, ..., k-1$, is a free abelian group of a finite rank.

Let $\varphi : N \to N$ be any automorphism. Then there are the induced automorphisms

$$\varphi_i : N_i \to N_i,\ \bar{\varphi}_i : A_i \to A_i,\ i = 0, 1, ..., k-1. \tag{14}$$

If $R(\varphi_i) = \infty$ for some $i = 0, 1, ..., k-1$, then $R(\varphi_j) = \infty$ for every $j < i$, in particular, $R(\varphi) = \infty$ in N. Moreover, if for some $i = 0, 1, ..., k-1$, there is a non trivial element $\bar{a} \in A_i$ such that $\bar{a}\bar{\varphi}_i = \bar{a}$, then $R(\varphi_j) = \infty$ for every $j < i$, and again $R(\varphi) = \infty$. To explain the last assertion we assume that i is maximal with the property $\bar{a}\bar{\varphi}_i = \bar{a} \neq 1$. Then the group N_{i+1} does not admit a non trivial element x such that $x\varphi_{i+1} = x$. Hence we have $L(N_i, \varphi_i) = L(A_i, \bar{\varphi}_i)$. Obviously $[A_i : L(A_i, \bar{\varphi}_i)] = \infty$. Then by Lemma 2.1 from [16] we derive that $R(\varphi_i) = \infty$ in N_i, and so $R(\varphi_j) = \infty$ for any $j < i$, in particular $R(\varphi) = \infty$.

Suppose that $R(\varphi) < \infty$. Then we have that the subgroup of φ_i-invariant elements $Fix_{\varphi_i}(N_i) = 1$ for all $i = 0, 1, ..., k-1$. We have $[A_i : L(N_i, \varphi_i)] = [A_i : L(A_i, \bar{\varphi}_i)] = q_i,\ i = 0, 1, ..., k-1$. Then we have

Lemma 2.1. *Let N be a finitely generated torsion free nilpotent group of class k, and $\varphi : N \to N$ be any automorphism. Suppose that $R(\varphi) < \infty$, and in notions as above $[A_i : L(N_i, \varphi_i)] = [A_i : L(A_i, \bar{\varphi}_i)] = q_i,\ i = 0, 1, ..., k-1$. Then*

$$R(\varphi) = \prod_{i=0}^{k-1} q_i. \tag{15}$$

Proof. The formula (15) is based on Lemma 2.4 from [16]. In this lemma a group G is considered with a central $\varphi-$admissible subgroup C for an automorphism $\varphi : G \to G$. It states that the pre image of any $\bar{\varphi}-$conjugacy class $[\bar{g}]_{\bar{\varphi}}$ of the induced automorphism $\bar{\varphi} : G/C \to G/C$ is a disjoint union of $s = [C : L(C, \varphi_g)]$ $\varphi-$conjugacy classes. Here $\varphi_g = \varphi \circ \sigma_g$, where $\sigma_g \in InnG, \sigma_g : h \mapsto g^{-1}hg$ for all $h \in G$.

We apply this statement consequently to groups $G = N_i$ and central subgroups $C = A_i, i = 0, 1, ..., k-1$. By our assumption $Fix_{\varphi_{i+1}}(N_{i+1}) = 1$, so if $x\varphi_i = cx, c \in A_i$, then $x \in A_i$. We see that $L(A_i, \varphi_i) = L(A_i, (\varphi_i)_g)$ for every $g \in N_i$. Hence every pre image of $[\bar{g}]_{\varphi_{i+1}}$ in N_i is a disjoint union of exactly (independent of g) $q_i = [A_i : L(A_i, \bar{\varphi}_i)]$ φ_i-conjugacy classes. It follows that

$$R(\varphi_i) = R(\varphi_{i+1}) \cdot q_i,\ i = 0, 1, ..., k-1. \tag{16}$$

Repeating such process we derive (15).

3 The Reidemeister specrum of N_{rc} for $r = 2$ and $c = 2, 3$; and $r = 3$ and $c = 2$

Let $N = N_{22}$ be the free nilpotent group of rank 2 and class 2 (also known as the discrete Heisenberg group). Let x, y be a free basis of N. Then the center $\zeta_1 N$ (which coincides with the derived subgroup N') is generated by a single basic commutator (x, y).

Let an automorphism $\varphi : N \to N$ induces the automorphism of abelianization $\bar{\varphi} : N/N' \to N/N'$, with matrix

$$A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, \tag{17}$$

matching to the basis x, y. We assume that $det(A - E) = trA = k \neq 0$, because in other case $R(\bar{\varphi}) = R(\varphi) = \infty$. Moreover, since $(x, y)\varphi = (x, y)^{det(A)}$ we assume that $det(A) = -1$, because $det(A) = 1$ implies $(x, y)\varphi = (x, y)$ and so $R(\varphi) = \infty$ again. Now we have $(x, y)\varphi = (x, y)^{-1}$, and so $[\zeta_1 N : L(N, \varphi)] = [\zeta_1 N : L(\zeta_1 N, \varphi)] = 2$. By Lemma 2.1 we obtain that $R(\varphi) = 2|trA| \in 2\mathbb{N}$. It remains to take the matrices $A(2, k)$ from the previous section to conclude that

$$Spec_R(N_{22}) = 2\mathbb{N} \cup \{\infty\}. \tag{18}$$

In paper [12] it was proved that each even number belongs to $Spec_R(N_{22})$, but nothing was said about odd numbers.

Let now $N = N_{23}$ be the free nilpotent group of rank 2 and class 3. Let x, y be a basis of N.

Then the center $\zeta_1 N$ has a basis consisting from the basic commutators of weight 3 : $g_1 = (x, y, x), g_2 = (x, y, y)$. Here and so far we suppose that the brackets in any long commutator stand from left to right, in particular $(h_1, h_2, h_3) = ((h_1, h_2), h_3)$, and so on.

It is well known (see [11]) that every automorphism of a free abelian quotient N/N' of any free nilpotent group N is induced by some automorphism of N itself. Moreover any endomorphism $\eta : N \to N$, invertible $mod(N')$ (inducing an automorphism in the abelianization N/N'), is automorphism. We will use this fact later many times without mention.

By direct calculation we derive that an automorphism $\varphi : N \to N$, with matrix on the abelianization N/N' as in (17) (we assume again that $det(A) = -1$ and $det(A - E) = trA \neq 0$) induces the automorphism $\bar{\varphi}_0 : \zeta_1 N \to \zeta_1 N$ with matrix in the basis g_1, g_2

$$A_3 = \begin{pmatrix} -\alpha & -\beta \\ -\gamma & -\delta \end{pmatrix}. \tag{19}$$

We see that

$$det(A_3) = -1, det(A_3 - E) = \alpha + \delta = trA. \tag{20}$$

Since every number $\mathbf{N}$ is obviously realized as $k = trA_{[k]}$ for some matrix $A_{[k]} \in GL_2(\mathbb{Z})$ we complete our consideration as above to conclude that

$$Spec_R(N_{23}) = \{2k^2 | k \in \mathbb{N}\} \cup \{\infty\}. \tag{21}$$

At last, let $N = N_{32}$ be the free nilpotent group of rank 3 and class 2. Let x, y, z be any basis of N. Then the center $\zeta_1 N$ (which coincides with the derived subgroup N') has a basis $h_1 = (x, y), h_2 = (x, z)$, and $h_3 = (y, z)$.

Let an automorphism $\varphi : N \to N$ induces the automorphism $\bar{\varphi} : N/N' \to N/N'$ with a matrix

$$A = (a_{ij}),\ i, j = 1, 2, 3, \tag{22}$$

in a basis of the abelianization N/N' matching to x, y, z. Then the matrix of the induced automorphism $\bar{\varphi_0} : N' \to N'$ is

$$B = \begin{pmatrix} M_{33} & M_{32} & M_{31} \\ M_{23} & M_{22} & M_{21} \\ M_{13} & M_{12} & M_{11} \end{pmatrix}, \tag{23}$$

where M_{ij} means the minor deriving by deleting $i-$row and $j-$column of B.

We can assume that $det(A - E) \neq 0$, in other case $R(\varphi) = \infty$. By direct calculation we get $det(B) = 1$. Also by direct calculation we obtain

$$det(A - E) = detA - (M_{11} + M_{22} + M_{33}) + a_{11} + a_{22} + a_{33} - 1, \tag{24}$$

and

$$det(B - E) = det(B) - det(A)(a_{11} + a_{22} + a_{33}) + M_{11} + M_{22} + M_{33} - 1. \tag{25}$$

We see that these numbers in view $det(A) = \pm 1$ and $det(B) = 1$ have the same parity. If they both are even then their product is divided by 4, if both are odd this product is odd too. It follows that numbers $4l + 2$ can not appear.

On the other hand a matrix

$$D_{[n]} = \begin{pmatrix} n & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \tag{26}$$

gives an example of automorphism $\varphi(n)$ which induces the automorphism of abelianization N/N' with matrix $D_{[n]}$ gives an example of $R(\varphi_n) = 2n - 1$.

A matrix

$$F_{[n]} = \begin{pmatrix} n+1 & 1 & 1 \\ 2 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \tag{27}$$

in the same way presents an example of an automorphism $\psi(n)$ for which $R(\psi(n)) = 4n$.

It follows that

$$Spec_R N_{23} = \{2n-1 | n \in \mathbf{N}\} \cup \{4n | n \in \mathbf{N}\} \cup \{\infty\}. \tag{28}$$

4 Every free nilpotent group N_{2c} for $c \geq 4$ satisfies to the R_∞ property

The main result of this section will follow from the next principal statement.

Theorem 1. Let $N = N_{24}$ be the free nilpotent group of rank 2 and class 4. Then every automorphism $\varphi : N \to N$ induces the automorphism $\bar{\varphi_0} : \zeta_1 N \to \zeta_1 N$ such that $det(\bar{\varphi_0} - E) = 0$.

Proof. Let x, y be a basis of N. Then the basic commutators of the weight 4 $f_1 = (x, y, x, x)$, $f_2 = (x, y, y, x)$, and $f_3 = (x, y, y, y)$ present a basis of $C = \zeta_1 N$ (see [11]). Since N is metabelian one has identity

$$(g, f, h_1, h_2) = (g, f, h_2, h_1). \tag{29}$$

Now suppose that any automorphism $\varphi : N \to N$ induces the automorphism of the abelianization $\bar{\varphi} : N/N' \to N/N'$ with a matrix (17) in the basis matching to x, y. So, $det(A) = \pm 1$. We assume that $det(A) = -1$, in other case $(x, y)\varphi = (x, y) mod \zeta_2 N$, and so $R(\varphi) = \infty$. Moreover, we can assume that $tr(A) \neq 0$, by a similar reason (see previous section). Let $\bar{\varphi_0} : \zeta_1 N \to \zeta_1 N$ be the automorphism induced by φ.

By direct calculation we define the matrix of $\bar{\varphi_0}$ in the basis f_1, f_2, and f_3 :

$$B = \begin{pmatrix} -\alpha^2 & -2\alpha\beta & \beta^2 \\ -\alpha\gamma & -\alpha\delta - \beta\gamma & -\beta\delta \\ -\gamma^2 & -2\gamma\delta & -\delta^2 \end{pmatrix}. \tag{30}$$

By direct calculation we derive that $det(B) = -1$, and $det(B - E) = 0$.

Since by our assumptions $Fix_{\varphi_1}(N_1) = 1$, where as above $N_1 = N/\zeta_1 N$ is the free nilpotent group N_{23} of rank 2 and class 3 (see previous section), we conclude that $L(N, \varphi) = L(\zeta_1 N, \bar{\varphi_0})$ has infinite index in $\zeta_1 N$. Hence by Lemma 2.4 from [16] we obtain that $R(\varphi) = \infty$.

Theorem is proved.

Corollary 4.1. Every group N_{2c} for $c \geq 4$ satisfies to the R_∞ property.

Since N_{24} is metabelian and in view of Theorem 2 in [16] we immediately obtain

Theorem 2. 1) Every free solvable group S_{2t} of rank 2 and class $t \geq 2$ (in particular, the free metabelian group $M_2 = S_{22}$ of rank 2) satisfies to the R_∞ property.

2) Every free solvable group S_{3t} of rank 3 and class $t \geq 4$ satisfies to the R_∞ property.

3) Every free solvable group S_{rt} of rank $r \geq 4$ and class $t \geq log_2(2r+1)$ satisfies to the R_∞ property.

Proof. We have for every $t \geq 2$

$$N_{24} = S_{2t}/\gamma_5 S_{2t}, \tag{31}$$

where $\gamma_5 S_{2t}$ is an automorphic admissible subgroup of S_{2t}. Any automorphism $\tilde{\varphi} : S_{2t} \to S_{2t}$ induces the automorphism $\varphi : N \to N$ for which $R(\varphi) = \infty$. Hence $R(\tilde{\varphi}) = \infty$ too.

2) By Theorem 2 in [16] one has $N_{3,12} \in R_\infty$. Since for every $t \geq 4$

$$N_{3,12} = S_{3t}/\gamma_{13} S_{3t}, \tag{32}$$

by the similar argument we conclude that S_{3t} satisfies to the R_∞ property.

3) By Theorem 2 in [16] one has $N_{r,2t+1} \in R_\infty$. Since for every $t \geq log_2(2r+1)$

$$N_{r,2r} = S_{rt}/\gamma_{2r+1} S_{rt}, \tag{33}$$

we again derive $S_{rt} \in R_\infty$.

Theorem is proved.

Remark. The similar results can be proved for any varieties of groups (not just the varieties $\mathcal{A}^t$ of all solvable groups of given class t) which admit a natural homomorphisms onto free nilpotent groups of class big enough.

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